April 21st 2016 **Name (Please Print)** _____

Probability II- Final Exam - **Maximum Time: 3 hours** **Maximum Score: 100**

1. (10 points) Suppose that X and Y are independent random variables both uniformly distributed in the interval $(0, 1)$. Suppose we create a right triangle for which two sides have lengths X and Y respectively. Let Z be the length of the hypotenuse. What is the probability that Z is larger than 1?

2. (15 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. Exponential (1) random variables. Let

$$T = \min\{k \in \mathbb{N} : X_1 + X_2 + \ldots + X_k > 3\}.$$

Find the distribution of T.

3. (10 points) Let $D \subset \mathbb{R}^2$ be the triangle bounded by the lines $y = 0$, $y = 1 - x$, and $y = 1 + x$. Suppose a random vector (X, Y) has a joint probability density function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_{(X,Y)}(x, y) = \begin{cases} 3y & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

Compute $E[Y|X = \frac{1}{2}]$.

4. (15 points) The institute has two borewells that supply water to campus. Suppose the amount of water (litres) from the two borewells on a given day are jointly bivariate normal with means 150 and 200, variances 100 and 25, and correlation .5.

   (a) What is the probability that the total amount of water supplied on any given day is larger than 400 litres?

   (b) What is the probability that the amount of water from the two wells on any day differ by more than 50 litres?

5. Let $X_1, X_2, \ldots, X_n$ be i.i.d Exponential (1) random variables and $U_1, U_2, \ldots U_n$ be i.i.d Uniform $(0, 1)$.

   (a) (10 points) For $1 \leq k \leq n$, let $Y_k = \sum_{i=1}^{k} X_i$. Find the joint density of $(Y_1, Y_2, \ldots, Y_n)$.

   (b) (10 points) Let $Z_n = \prod_{i=1}^{n} U_i$. Using (a), find the probability density function of Z_n.

6. (10 points) Let $X \sim$ Exponential (λ). Find the characteristic function of X.

7. (20 points) We wish to find the probability of heads of a biased coin. Suppose we know that the probability of heads $p \in (0.6, 0.9)$. How many independent and identical tosses of the coin are needed in order that we can be 95% sure that the observed frequency differs from p by less than $p/10$?

Note: Normal tables evaluating : $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{x^2}{2}} dx$ *are on the back page.*

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2882	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3079	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3290	0.3315	0.3340	0.3365	0.3389
1.0	0.3414	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3622
1.1	0.3643	0.3665	0.3687	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4083	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4193	0.4207	0.4222	0.4237	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4358	0.4370	0.4382	0.4394	0.4406	0.4418	0.4430	0.4441
1.6	0.4452	0.4463	0.4474	0.4485	0.4495	0.4505	0.4516	0.4526	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4600	0.4608	0.4617	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4679	0.4686	0.4693	0.4700	0.4706
1.9	0.4713	0.4720	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4762	0.4767
2.0	0.4773	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4813	0.4817
2.1	0.4822	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4858
2.2	0.4861	0.4865	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4914	0.4916
2.4	0.4918	0.4920	0.4923	0.4925	0.4927	0.4929	0.4931	0.4933	0.4934	0.4936
2.5	0.4938	0.4940	0.4942	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4954	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4966	0.4967	0.4968	0.4969	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4975	0.4975	0.4976	0.4977	0.4978	0.4978	0.4979	0.4980	0.4980	0.4981
2.9	0.4982	0.4982	0.4983	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4991	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4993	0.4993	0.4993
3.2	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995	0.4995
3.3	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998	0.4998	0.4998

Table 1: Normal tables evaluating : $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{x^2}{2}} dx$